Also

$$t_{lag} = \lim (\psi_0/\omega)_{\omega \to 0} = \lim \left(\frac{d\psi}{d\omega} \right)_{\omega \to 0} =$$

$$- [(\rho c)_s/(\rho c)_g - \epsilon ((\rho c)_s/(\rho c)_g - 1)] \, L/\epsilon' U \quad (B5)$$

$$\approx - (\rho c)_s \, (1 - \epsilon) \, L/(\rho c)_g \, \epsilon' U \quad (B6)$$

[The approximate forms, Equations ($B4$) and ($B6$), are permitted if $(\rho c)_s/(\rho c)_g \approx 1{,}000$]. From the definition of Z it follows from Equation ($B6$) that $Z \approx 3t_{lag}/r_0 \, (\rho c)_s$. Again

$$\lim (\psi)_{\omega \to \infty} = \lim (\Pi)_{\omega \to \infty} = \lim [(s/2)^{1/2}]_{\omega \to \infty} = -(\epsilon \omega L^2/2D\epsilon')^{1/2} \quad (B7)$$

and

$$\lim \left(\frac{d\psi}{d(\omega^{1/2})} \right)_{\omega \to \infty} = \left(\frac{d\Pi}{d(\omega^{1/2})} \right)_{\omega \to \infty} = - L(\epsilon/\epsilon' D)^{1/2}/2 \quad (B8)$$

[In principle Equation ($B8$) could be used for estimating D.]

Manuscript received April 25, 1966; revision received October 26, 1966; paper accepted October 26, 1966.

Generalized Solution of the Tomotika Stability Analysis for a Cylindrical Jet

BERNARD J. MEISTER and GEORGE F. SCHEELE

Cornell University, Ithaca, New York

The stability of cylindrical jets in immiscible liquid systems is analyzed with the low velocity theory of Tomotika. For the first time the several limiting solutions in the literature are obtained from a general equation, so approximate restrictions on their applicability can be presented. These restrictions show that for many systems none of the limiting solutions is valid. Correlations applicable to all Newtonian liquid-liquid systems are presented for predicting the growth rate and wavelength of the most unstable disturbance.

The injection of one liquid into another is important in many industrial operations. At low injection velocities drops are formed directly at the nozzle and their size is controlled by the forces acting on the forming drop (*3*). At higher injection velocities a jet of liquid issues from the nozzle and then breaks into droplets in a regular pattern. This breakup of a cylinder of liquid has interested many scientists.

In 1873 Plateau (*6*) showed that a cylinder of liquid subject to surface forces is unstable if its length exceeds its circumference, because it can be divided into two spheres of equal volume with an accompanying decrease in surface area. This analysis indicated that surface forces are the cause of jet breakup and that the waves visible on the jet surface should have a wavelength equal to the circumference of the jet.

In 1879 Lord Rayleigh (*7, 8*) set forth several postulates concerning wave forms on a jet which have been the basis of most subsequent instability analyses. He assumed that disturbances corresponding to all possible wavelengths are initiated at the nozzle exit as a result of density and pressure fluctuations. The amplitude ξ of any resulting wave is given by the equation

$$\xi = \xi_0 \exp (\alpha t + ikz) \quad (1)$$

where ξ_0 is the initial amplitude of the disturbance and λ is the disturbance wavelength which is related to the wave number k by the equation

$$\lambda = 2\pi/k \quad (2)$$

Bernard J. Meister is with the Dow Chemical Company, Midland, Michigan.

All waves with a wavelength greater than the jet circumference have a positive growth rate α and amplify with time. Rayleigh assumed that all the initial disturbances are very small and of the same magnitude. Therefore the wave with the largest growth rate becomes the dominant wave on the jet surface and ultimately breaks the jet into drops. Rayleigh derived equations predicting the disturbance growth rate for a nonviscous liquid jet in a gas (*7, 8*), a gas jet in a nonviscous liquid (*10*), and an extremely viscous liquid in a gas (*9*). Further derivations have been made by Weber (*14*) for a viscous liquid jet in a gas, by Tomotika (*13*) for a very viscous liquid jet in a very viscous liquid medium, and by Christiansen (*1, 2*) for a nonviscous liquid jet in a nonviscous liquid medium. Other relevant theoretical instability studies have included non-Newtonian and viscoelastic effects (*5, 16, 17*) and jet contraction effects (*15*). All these equations are limited to jet velocities sufficiently low that the gross jet velocity does not affect the wavelength and growth rate of the dominant disturbance.

The agreement between experimental results and theoretical instability analyses has been good when experiments are performed which closely approximate the assumptions made in the theoretical development (*1, 4, 11, 12*). There are, however, two significant limitations to these analyses which prevent direct application to many experimental studies. In the first place, while the solutions are for limiting cases, all of the analyses except Tomotika's assume the character of the fluids at the start of the derivation, so it is not possible to set quantitative limits on the applicability of the resulting equations. This has created uncertainty as to which analysis should be applied in a particular situation because there are no estimates available of the ranges of physical properties over which each limiting solution will give satisfactory predictions of the growth rate and wavelength. Second, there are many systems which are not well approximated by any of the limiting solutions.

This paper uses Tomotika's general low velocity equations as the starting point for deriving the limiting solutions. In this way it is possible to establish quantitatively the regions of applicability of these solutions. Generalized correlations are also presented for the dominant disturbance growth rates and wavelengths which are applicable to all liquid-liquid systems including those for which none of the limiting solutions provides a good approximation.

LIMITING SOLUTIONS AND REGIONS OF APPLICABILITY

Tomotika derived a general low velocity equation relating the growth rate of a disturbance to liquid properties, jet diameter, and wavelength of the disturbance by writing the equations of motions for an incompressible Newtonian fluid for both phases and relating the two boundary conditions at the interface (*13*). The three main assumptions in his analysis are: (1) The momentum balances for the gross motion of the jet and the perturbation are independent. (2) The perturbation is symmetrical with respect to the jet axis. (3) Terms which are second order in the perturbation are negligible.

It should be emphasized that Tomotika's theory is strictly applicable only when the jet and surrounding fluid are in uniform rectilinear flow. In actual capillary jets the relative motion of the two phases becomes increasingly important in determining stability characteristics as the relative velocity increases.

Tomotika's general equation [Equation (33) in reference *13*] is the basis for the present work. All of the well-known limiting solutions for Newtonian fluids can be derived from this general equation. A detailed presentation of these solutions is given by Meister (*3*).

CASE 1: LOW VISCOSITY LIQUID JET IN A GAS

The continuous phase density and viscosity terms and the dispersed phase viscosity terms can be neglected. For a low viscosity liquid it can further be assumed that

$$\frac{\alpha\rho'}{\mu'} >> k^2 \tag{3}$$

The primed quantities refer to the dispersed or jet phase. The general equation then simplifies to

$$\alpha^2 = \frac{\sigma(1-k^2a^2)ka}{\rho' a^3 I_0(ka)/I_1(ka)} \tag{4}$$

This is identical to the equation derived by Rayleigh (*7*), and the wavelength with maximum growth rate corresponds to the dimensionless wave number $(ka)_{\max} = 0.696$.

The continuous phase viscosity and density terms are extremely small when the continuous phase is a gas, so no additional restriction need be placed on these variables. The only other limitation on Equation (4) is that imposed by Equation (3). After substitution of Equation (4) for α and 0.696 for ka, Equation (3) becomes

$$\frac{(\sigma\rho' D_N)^{1/2}}{\mu'} >> 2.0 \tag{5}$$

A criterion for elimination of terms must be selected. In this and the following limiting solutions, a 5% error in α was considered acceptable. Numerical substitution into the general equation showed that the neglected terms must be less than 10% of the retained terms. The resulting restriction on the application of Equation (4) to liquid jets in gases is

$$\frac{(\sigma\rho' D_N)^{1/2}}{\mu'} > 20 \tag{6}$$

Typical values of 20 dynes/cm. for the surface tension, 1.0 g./cc. for the liquid density, and 0.20 cm. for the nozzle diameter were employed in this and the following limiting solutions to obtain an estimate of the requirement on the liquid viscosity. With these values, Equation (6) requires that the jet viscosity be less than 10 centipoise.

CASE 2: GAS JET IN A LOW VISCOSITY LIQUID

The density and viscosity terms for the jet phase and the continuous phase viscosity terms can be neglected. For a low viscosity continuous liquid phase, it can also be assumed that

$$\frac{\alpha\rho}{\mu} >> k^2 \tag{7}$$

The general equation then simplifies to

$$\alpha^2 = \frac{\sigma(1-k^2a^2)ka}{\rho\, a^3 K_0(ka)/K_1(ka)} \tag{8}$$

This is identical to the equation derived by Rayleigh (*10*) for gas jets in nonviscous liquids and the controlling wavelength corresponds to the dimensionless wave number $(ka)_{\max} = 0.485$. The continuous phase viscosity terms are the largest of those neglected. If the same procedure as in case 1 is used, the restriction on the applicability of Equation (8) is

$$\frac{(\sigma\, \rho\, D_N)^{1/2}}{\mu} > 36 \tag{9}$$

With the same representative values for the nozzle diameter and liquid properties, the continuous phase viscosity must be less than 6 centipoise.

CASE 3: LOW VISCOSITY LIQUID JET IN A LOW VISCOSITY LIQUID

The viscous terms for both phases can be neglected. If it is further assumed that $k << m$ and $k << m'$, where

$$m^2 = k^2 + \frac{\alpha\rho}{\mu} \tag{10}$$

$$m'^2 = k^2 + \frac{\alpha\rho'}{\mu'} \tag{11}$$

the general equation simplifies to

$$\alpha^2 = \frac{\sigma(1-k^2a^2)ka}{a^3\,[\rho' I_0(ka)/I_1(ka) + \rho\, K_0(ka)/K_1(ka)]} \tag{12}$$

This is identical to the equation derived by Christiansen (*1*) for nonviscous liquid jets in a nonviscous liquid. The wave number of the dominant wave is given in Figure 1 as a function of the density ratio. With the substitution of the solutions for α and k back into the assumptions, the criteria for a maximum 5% error in growth rate become

$$\frac{[\sigma\rho^2 D_N/(3.15\rho' + 0.62\rho)]^{1/2}}{\mu} > 103 \tag{13}$$

and

$$\frac{[\sigma\rho'^2 D_N/(3.15\rho' + 0.62\rho)]^{1/2}}{\mu'} > 103 \tag{14}$$

For the typical values of liquid properties and nozzle diameter, Equations (13) and (14) require that both the

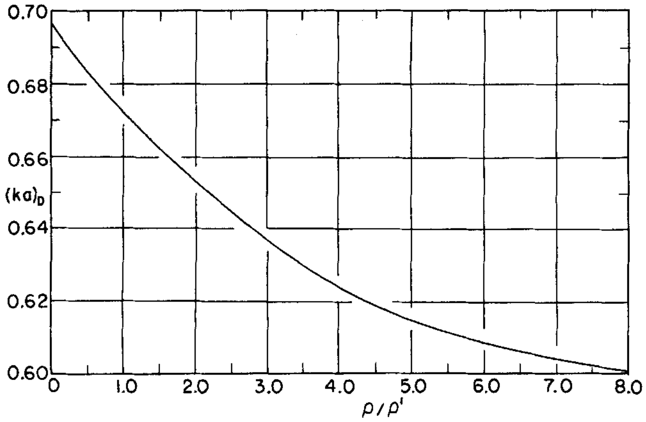

Fig. 1. Values of *ka* which maximize the disturbance growth rate for a low viscosity liquid jet in a low viscosity liquid medium.

continuous and dispersed phase viscosities be less than 1.0 centipoise.

CASE 4: HIGH VISCOSITY LIQUID JET IN A GAS

The density and viscosity terms for the continuous phase can be neglected. Also, when $\mu'/\mu >> 1$, $(ka)_{\max}$ becomes small and the Bessel functions in the general equation can be approximated by their limiting values as the argument approaches zero.

$$\frac{I_1'(m'a)}{I_1(m'a)} \approx \frac{1}{m'a} \tag{15a}$$

$$\frac{I_1'(ka)}{I_1(ka)} \approx \frac{1}{ka} \tag{15b}$$

and

$$\frac{I_0(ka)}{I_1(ka)} \approx \frac{2}{ka} \tag{16}$$

Use of these approximations yields

$$2\,\rho'\,\alpha^2\,\frac{(k^2+m'^2)}{(m'^2-k^2)} + 2\,\mu'\,k^2\,\alpha = \frac{\sigma(1-k^2a^2)k^2a^2}{a^3} \tag{17}$$

Substitution of Equation (11) for m' into Equation (17) gives

$$\alpha^2 + \frac{3\mu' k^2 \alpha}{\rho'} = \frac{\sigma(1-k^2a^2)k^2a^2}{2\,\rho'\,a^3} \tag{18}$$

Equation (18) is identical to Weber's equation for viscous liquids in gases and, as shown by Weber (*14*), the dominant wavelength is given by

$$(ka)_{\max} = \frac{1}{1.415\left(1+\dfrac{3\mu'}{\sqrt{2\,\rho'\,\sigma\,a}}\right)^{1/2}} \tag{19}$$

The restricting assumption is Equation (15*a*) which requires that

$$\frac{\alpha\rho'}{\mu'} < k^2 \tag{20}$$

Substitution of the α and k values associated with the dominant wave yields the requirement that

$$\frac{(\sigma\rho' D_N)^{1/2}}{\mu'} < 1.20 \tag{21}$$

For most systems Equation (21) requires a jet viscosity greater than 160 centipoise, which is not noted by most investigators when they apply the Weber equation.

CASE 5: HIGH VISCOSITY LIQUID JET IN A LOW VISCOSITY LIQUID

The assumptions required are identical to those in case 4, except that the continuous phase density is not negligible. However, the resulting equation shows the continuous phase density terms to be small when

$$\rho < 6\rho' \tag{22}$$

When both phases are liquids, Equation (22) is generally satisfied. The general equation then reduces to Equation (18), which was derived for high viscosity liquids in gases, and the dominant wavelength is given by Equation (19). For a liquid continuous phase, neglect of the continuous phase viscosity terms adds the restriction that

$$\frac{\alpha\rho}{\mu} > 100k^2 \tag{23}$$

By substituting the α and k values for the fastest growing wave, determined from Equations (18) and (19), and noting that

$$\frac{3\mu'}{(\rho'\sigma D_N)^{1/2}} >> 1 \tag{24}$$

the resulting restriction on the continuous phase viscosity is

$$\frac{(\sigma\,\rho\,D_N)^{1/2}}{\mu} > 6{,}667\,\frac{\mu}{\mu'}\left(\frac{\rho'}{\rho}\right)^{3/2} + 300\left(\frac{\rho'}{\rho}\right)^{1/2} \tag{25}$$

For the representative liquid properties Equation (21) requires that the jet viscosity be greater than 160 centipoise, and for a jet viscosity of 160 centipoise, Equation (25) requires that the continuous phase viscosity be less than 0.60 centipoise.

CASE 6: LOW VISCOSITY LIQUID JET IN A HIGH VISCOSITY LIQUID

The viscous terms for the dispersed phase can be neglected. For a very high viscosity continuous phase, the values of ka and ma are small and nearly equal, so that

$$\frac{K_1'(ma)}{K_1(ma)} \approx \frac{K_1'(ka)}{K_1(ka)} \tag{26}$$

and

$$\frac{K_0(ma)}{K_1(ma)} \approx \frac{K_0(ka)}{K_1(ka)} \tag{27}$$

In addition it can be shown that

$$\frac{I_0(m'a)}{I_1(m'a)}\,m' >> \frac{K_0(ka)}{K_1(ka)}\,k \tag{28}$$

The general equation then reduces to

$$\rho'\alpha^2\,\frac{I_0(ka)}{I_1(ka)} + \rho\alpha^2\,\frac{K_0(ka)}{K_1(ka)}\left(\frac{m^2+k^2}{m^2-k^2}\right) - 2\mu k^2\alpha\,\frac{K_1'(ka)}{K_1(ka)} = \sigma\,\frac{(1-k^2a^2)ka}{a^3} \tag{29}$$

By substituting Equation (10) for m and noting that

$$K_1'(ka) = K_0(ka) - \frac{K_1(ka)}{ka} \tag{30}$$

and that for low values of ka, Equation (16) is valid, and that

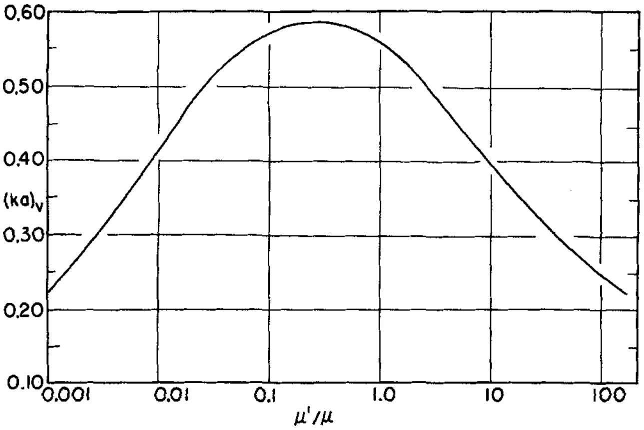

Fig. 2. Values of *ka* which maximize the disturbance growth rate for a high viscosity liquid jet in a high viscosity liquid medum.

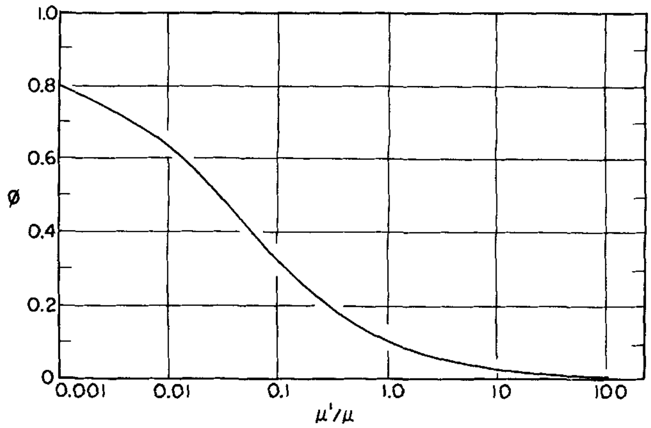

Fig. 3. Values of ϕ for use in Equation (38) to predict the disturbance growth rate for a high viscosity liquid jet in a high viscosity liquid medium.

$$\frac{K_0(ka)}{K_1(ka)} \to 0 \tag{31}$$

Equation (29) becomes

$$\alpha^2 + \frac{\mu k^2}{\rho'}\alpha = \frac{\sigma(1-k^2a^2)k^2a^2}{2\rho' a^3} \tag{32}$$

The dominant wavelength, obtained by differentiating Equation (32) with respect to k and setting $\partial\alpha/\partial k = 0$, is

$$\lambda_{max} = 2.83\pi a \left[1 + \frac{\mu}{\sqrt{2\rho'\sigma a}}\right]^{1/2} \tag{33}$$

There are several restrictions on Equation (32). The continuous phase density terms have been dropped, which is valid when Equation (22) is satisfied. The use of Equations (26) and (27) requires that

$$\frac{\alpha\rho}{\mu} < 0.10k^2 \tag{34}$$

Substitution of α and ka obtained by combining Equations (32), (2), and (33) yields the restriction that

$$\frac{\sigma\,\rho\,D_N}{\mu^2} < 0.02\left(\frac{\rho'}{\rho}\right)^{3/2} + 0.20\left(\frac{\rho'}{\rho}\right)^{1/2} \tag{35}$$

The use of Equation (28) further requires that

$$\frac{\alpha\rho'}{\mu'} > 100k^2 \tag{36}$$

which, after substitution of the expressions for α and ka, yields the restriction that

$$\frac{(\sigma\,\rho'\,D_N)^{1/2}}{\mu'} > 20{,}000\frac{\mu'}{\mu} + 200 \tag{37}$$

For the typical liquid properties employed in case 1, Equation (35) requires that the continuous phase viscosity be greater than 400 centipoise and for a liquid having this viscosity, Equation (37) requires that the jet viscosity be less than 0.80 centipoise.

CASE 7: HIGH VISCOSITY LIQUID JET IN A HIGH VISCOSITY LIQUID

This is the situation analyzed by Tomotika (*13*). By recognizing that for high viscosity phases $m \approx m' \approx k$, Tomotika was able to show that the dominant wave number $(ka)_{max}$ is a function only of the viscosity ratio. This dependence, reproduced in Figure 2 from Tomotika's original work, is also useful in establishing the generalized correlations presented in the last section of the paper.

The growth rate of the dominant disturbance can be expressed as

$$\alpha_{max} = \frac{\sigma}{2a\mu}(1-k^2a^2)\,\phi \tag{38}$$

where ϕ is a function of the viscosity ratio and is plotted in Figure 3. The restriction on the continuous phase viscosity for the application of Equation (38) is given by Equation (34). For large values of μ, it may be assumed that μ'/μ is small. For this limit $\phi = 1$ and substitution of Equation (38) into Equation (34) yields

$$\frac{\sigma\rho}{2a\mu^2}(1-k^2a^2) < 0.10k^2 \tag{39}$$

With the use of a typical value of $ka = 0.50$, Equation (39) becomes

$$\frac{\sigma\,\rho\,D_N}{\mu^2} < 0.133 \tag{40}$$

The restriction on the dispersed phase viscosity can be similarly obtained by using the approximation that at high μ'/μ, $\phi = \mu/3\mu'$. This restriction then becomes

$$\frac{\sigma\,\rho'\,D_N}{\mu'^2} < 0.40 \tag{41}$$

For the typical values of liquid properties and nozzle diameter, Equations (40) and (41) indicate that the continuous phase viscosity should be greater than 400 centipoise and the dispersed phase viscosity greater than 225 centipoise for the application of Equation (38). A summary of the limiting solutions, the corresponding equations for the wavelength and the growth rate of the dominant disturbance, and the range of applicability of the equations are presented in Table 1.

COMPARISON OF LIMITING SOLUTIONS WITH GENERAL COMPUTER SOLUTION

The special cases discussed and the approximate limits to their applicability demonstrate that the most widely used equations are very restricted in their application. For many liquid-liquid systems of practical importance, none of the solutions is valid. In general the growth rate and wavelength of the dominant disturbance are dependent on six variables: the jet diameter, the interfacial tension, and the densities and viscosities of the two phases. Tc motika's general equation was solved on a Control Dat 1604 digital computer to determine how each variabl and the interactions among the variables affect the growt rate and wavelength of the dominant disturbance. To solve the equation for a particular system requires an iterative

TABLE 1. SUMMARY OF EQUATIONS FOR PREDICTION OF THE DIMENSIONLESS WAVE NUMBER AND GROWTH RATE OF DOMINANT DISTURBANCE

Type of system: Dispersed phase	Continuous phase	Prediction of growth rate	Prediction of wave number	Limitations of use
Nonviscous liquid	Gas	(4)	$ka = 0.696$	(6)
Gas	Nonviscous liquid	(8)	$ka = 0.485$	(9)
Very viscous liquid	Gas	(18)	(19)	(21)
Nonviscous liquid	Nonviscous liquid	(12)	Figure 1	(13), (14)
Very viscous liquid	Nonviscous liquid	(18)	(19)	(21), (22), (25)
Nonviscous liquid	Very viscous liquid	(32)	(33)	(22), (35), (37)
Very viscous liquid	Very viscous liquid	(38) and Figure 3	Figure 2	(40), (41)
Liquid	Liquid	(12) and Figure 8	Figures 1 and 9	$\frac{\mu'}{\mu} > 10, \frac{\mu'}{\mu} < 0.01$
Liquid	Liquid	(49), (12), (38)	(44) and Figures 6 and 7	$0.01 < \frac{\mu'}{\mu} < 10$

procedure to determine the growth rate α for a given wave number ka, which must be repeated for values of ka between 0 and 1 until the wave number with the highest growth rate is determined.

Figures 4 and 5 show how the various limiting equations compare with the general solution. Figure 4 shows the effect of varying the dispersed phase viscosity on the growth rate of the controlling disturbance. At low viscosities the general solution approaches Equation (12), and at high viscosities it merges with Equations (18) and (38). Figure 5 shows the effect of varying the continuous phase viscosity on the growth rate of the controlling disturbance. In this case the general solution approaches Equation (12) at low viscosities and merges with Equations (32) and (38) at high viscosities. Similar plots could be drawn showing the effects of the other four variables.

The possible error involved by employing the limiting solutions can be illustrated for a liquid-liquid system having a jet phase of density 0.87 g./cc. and viscosity 10 centipoise issuing from a nozzle of 0.0813 cm. diameter into a continuous phase of density 1.0 g./cc. and viscosity 10 centipoise with an interfacial tension of 36.2 dynes/cm. between the two liquids. The two most likely limiting solutions for this system are the Christiansen equation (*12*) and Tomotika's Equation (38). Christiansen's equation predicts a dominant wave having a dimensionless wave number $ka = 0.670$ and a growth rate of 244 sec.$^{-1}$ Tomotika's equation predicts values of 0.565 and 220 sec.$^{-1}$, respectively. The values obtained from the general computer solution are $ka = 0.640$ and $\alpha = 141.1$ sec.$^{-1}$, so the two most reasonable limiting solutions have errors of greater than 50% in the growth rate of the controlling disturbance.

GENERALIZED CORRELATION FOR LIQUID-LIQUID SYSTEMS

Tables could be developed which would give the values of growth rate and wavelength of the controlling disturbance for any set of the six variables. This would be an exhaustive procedure, so for practical purposes an approximate correlation has been developed.

For liquid-liquid systems the two limiting cases which are of most importance in developing a correlation for all systems are Equation (12), which neglected all viscous effects, and Equation (38), which neglected all inertial effects. A correlating parameter is required which will predict where, between these two extremes, a particular system lies. The parameter which is suggested by the re-

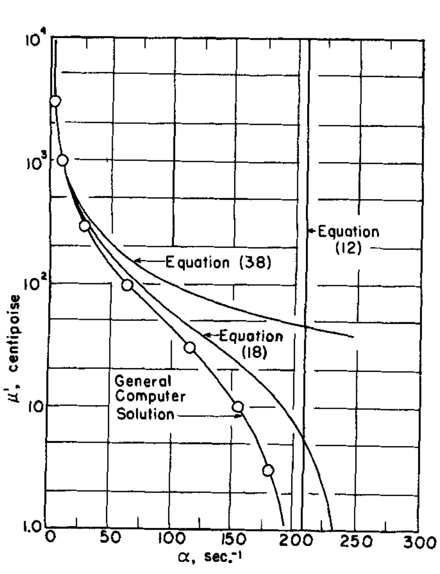

Fig. 4. Comparison of limiting solutions with the general jet instability solution as a function of dispersed phase viscosity. D_N = 0.0813 cm., ρ' = 0.870 g./cc., ρ = 1.0 g./cc., μ = 0.01 g./(cm.)(sec.), σ = 26.2 dynes/cm.

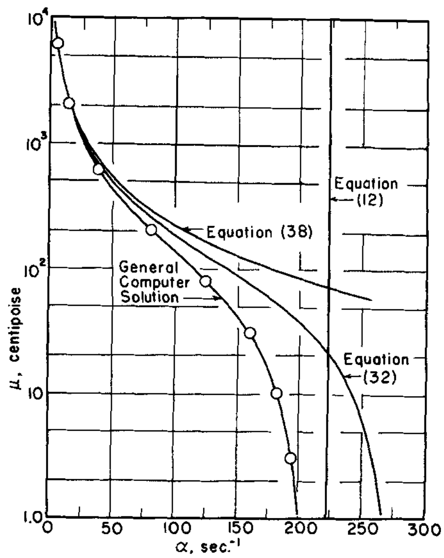

Fig. 5. Comparison of limiting solutions with the general jet instability solution as a function of continuous phase viscosity. D_N = 0.0813 cm., ρ' = 0.680 g./cc., ρ = 1.254 g./cc., μ' = 0.004 g./(cm.)(sec.), σ = 26.2 dynes/cm.

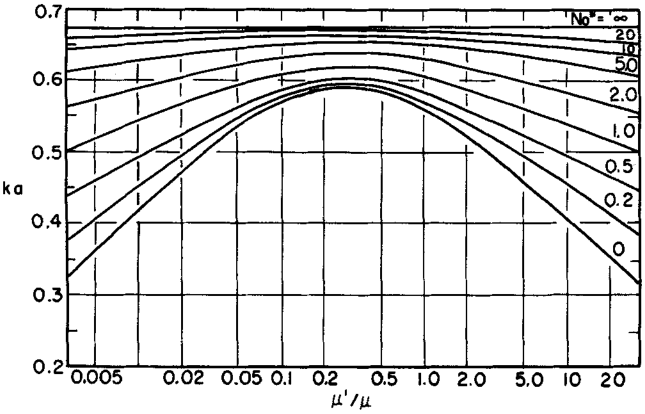

Fig. 6. Wave number correlation for liquid-liquid systems with $0.01 < \mu'/\mu < 10.0$ and $\rho/\rho' = 1.0$.

strictions on the limiting solutions is the Ohnesorge number.

$$N_0 = \frac{(\sigma \rho D_N)^{1/2}}{\mu} \quad N_0' = \frac{(\sigma \rho' D_N)^{1/2}}{\mu'} \tag{42}$$

Two additional facts can be obtained from the limiting solutions. The dispersed phase density is much more important than the continuous phase density and the dispersed phase viscosity is approximately three times as important as the continuous phase viscosity. These observations led to the choice of a correlating parameter N_0^*, defined by

$$N_0^* = \frac{(\sigma \rho' D_N)^{1/2}}{\mu + 3\mu'} \tag{43}$$

It was not possible to develop correlations for the wave number and growth rate which were accurate for all possible liquid-liquid systems, so the multivariate surface was divided into two regions, one where the viscosity ratio μ'/μ is between 0.01 and 10.0, and one for both higher and lower values of the viscosity ratio.

Moderate Viscosity Ratios

The wave number correlation for viscosity ratios between 0.01 and 10.0 is shown in Figure 6 for a density ratio $\rho/\rho' = 1.0$. The wave number ka for the fastest growing disturbance is plotted vs. the viscosity ratio with N_0^* as a parameter. The solutions for $N_0^* = \infty$ and $N_0^* = 0$ correspond to limiting cases 3 and 7, respectively. Computer solutions were used to obtain the curves for intermediate values of N_0^*.

If the density ratio is not 1.0, a correction factor must be applied. Because the dependence on density ratio is weak and liquid densities are usually between 0.50 and 2.0 g./cc., the correction factor is small. A plot of the correction factor as a function of ρ/ρ' and N_0^* is presented in Figure 7 for liquid-liquid systems having density ratios different from 1.0. Knowing the values of μ'/μ, ρ/ρ' and N_0^* for a given system, one can determine the controlling wave number from Figures 6 and 7 by using the equation

$$(ka)_{\max} = (ka)_{\frac{\rho}{\rho'}=1} + C_0 \tag{44}$$

To determine the growth rate α, it is convenient to rewrite Tomotika's general equation as

$$X\alpha^2 + Y\alpha = T \tag{45}$$

where X and Y are complicated functions. X is zero when the density terms are neglected, and α is given by Equation (38). This α, which will be denoted by α_v, can also be obtained from Equation (45) and is

$$\alpha_v = \frac{T}{Y} \tag{46}$$

Y is zero when the viscosity terms are neglected, and α is given by Equation (12). This α will be denoted by α_D. Solution of Equation (45) for α_D gives

$$\alpha_D = \left(\frac{T}{X}\right)^{1/2} \tag{47}$$

The general solution to Equation (45) is

$$\alpha = -\frac{Y}{2X} + \left(\frac{Y^2}{4X^2} + \frac{T}{X}\right)^{1/2} \tag{48}$$

which can be written in terms of α_v and α_D by substitution of Equations (46) and (47).

$$\alpha = -\frac{\alpha_D^2}{2\alpha_v} + \left(\frac{\alpha_D^4}{4\alpha_v^2} + \alpha_D^2\right)^{1/2} \tag{49}$$

An approximate solution for α can be obtained from Equation (49) by substituting α_D, calculated from Equation (12) using Figure 1 for ka, and α_v, calculated from Equation (38) using Figure 2 for ka. The method is not mathematically exact because different approximations were made in obtaining Equations (12) and (38), and the values of ka used in the two equations are not the same. However, the values of α computed from Equation (49) with this approach show good agreement with the numerical solution. For the system discussed previously where the viscosity of each phase was 10 centipoise, the predictions of the correlation are $(ka)_{\max} = 0.640$ and $\alpha = 143$ which agree well with the computer solutions of 0.640 and 141.1.

Extreme Viscosity Ratios

Although the correlations for moderate viscosity ratios could be extended to very high or very low viscosity ratios, their accuracy diminishes. Much simpler correlations can be used in these regions as the viscosity of only one of the phases is important.

The starting point for the correlations is the limiting solution for case 3. If $(ka)_D$ and α_D represent the wave number and growth rate of the most unstable wave when both phases have a low viscosity, it is found that to a good approximation the ratios $ka/(ka)_D$ and α/α_D for systems having extreme viscosity ratios are functions only of N_0^*, because interactions between the viscous terms for the two phases are negligible. These functions are shown in Figures 8 and 9 together with representative points from the general computer solution. To use the correlations $(ka)_D$ is obtained from Figure 1 and α_D is calculated from Equation (12) using the determined $(ka)_D$ value.

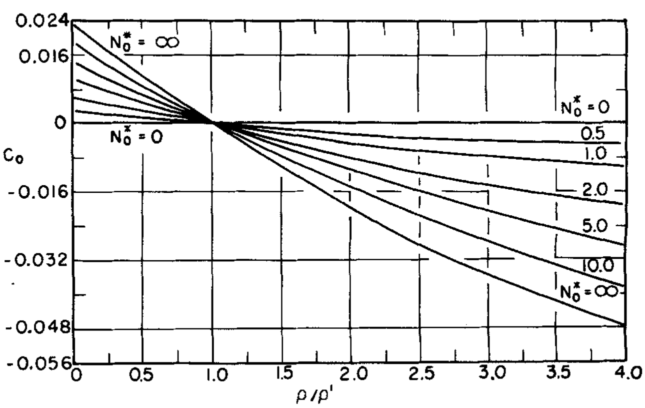

Fig. 7. Correction factor for use with Figure 6 in Equation (44) when the density ratio is not 1.0.

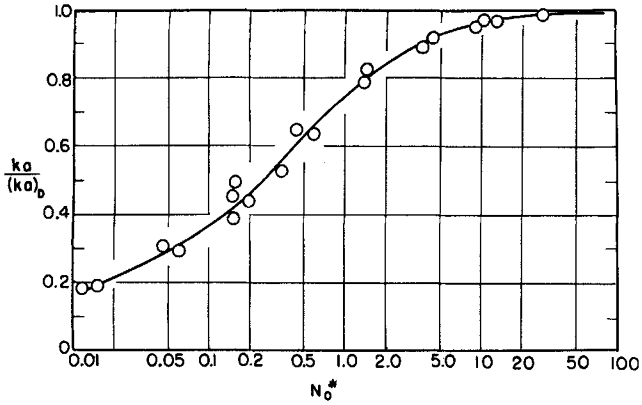

Fig. 8. Wave number correlation for liquid-liquid systems with $\mu'/\mu > 10.0$ or $\mu'/\mu < 0.01$. The points represent the computer solution to Tomotika's general equation.

Predicted wavelengths and growth rates cannot be compared directly with experimental data for the injection of one liquid into another. By the time a wave has amplified sufficiently to be observed photographically, the jet has deformed to a noncylindrical shape. However, the values of the wavelength and growth rate of the dominant wave are important for predicting jet length and drop size and use of the developed correlations has significantly improved these predictions, as will be discussed in a future paper.

SUMMARY

Limiting solutions for jet stability can often be used to predict the wavelength and growth rate of the most unstable disturbance. The present paper contains the first quantitative criteria for determining whether a limiting solution is applicable. For those situations where no limiting solution is valid, useful correlations have been established from a numerical solution to Tomotika's stability analysis.

The present calculations are based on a model which assumes that both fluids are everywhere moving at the same uniform velocity. There is apparently a lower range of injection velocities for liquid-liquid systems for which this model is valid. However, for higher injection velocities the influence on stability of nonuniform velocity distributions associated with viscous shear becomes important and must be considered to explain experimental phenomena. A future paper will consider this problem in more detail and will present criteria concerning the applicability of the present analysis.

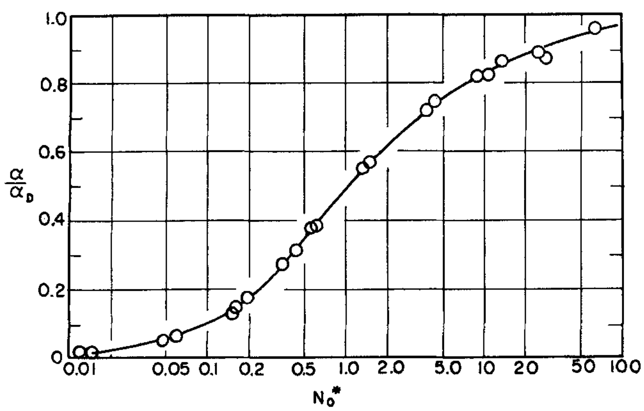

Fig. 9. Disturbance growth rate correlation for liquid-liquid systems with $\mu'/\mu > 10.0$ or $\mu'/\mu < 0.01$. The points represent the computer solution to Tomotika's general equation.

ACKNOWLEDGMENT

The authors gratefully acknowledge the Ford Foundation Fellowship awarded to B. J. Meister.

NOTATION

a = jet radius, cm.
C_0 = correction factor for density ratios other than 1.0
D_N = nozzle diameter, cm.
i = square root of -1
$I_n(x)$ = modified Bessel function of the first kind of order n
$I_n'(x)$ = derivative of $I_n(x)$
k = wave number of disturbance, cm.$^{-1}$
ka = dimensionless wave number
$K_n(x)$ = modified Bessel function of the second kind of order n
$K_n'(x)$ = derivative of $K_n(x)$
m = parameter defined by Equation (10)
m' = parameter defined by Equation (11)
N_0 = Ohnesorge number, $(\sigma\rho D_N)^{1/2}/\mu$
N_0' = Ohnesorge number, $(\sigma\rho' D_N)^{1/2}/\mu'$
N_0^* = modified Ohnesorge number, $(\sigma\rho' D_N)^{1/2}/(3\mu' + \mu)$
t = time, sec.
T, X, Y = dummy variables employed in Equation (45)
z = axial distance, cm.

Greek Letters

α = growth rate of disturbance, sec.$^{-1}$
λ = wavelength of disturbance, cm.
μ, μ' = viscosities of continuous and dispersed phases, respectively, g./(cm.)(sec.)
ξ = amplitude of disturbance, cm.
ξ_0 = initial amplitude of disturbance, cm.
ρ, ρ' = densities of continuous and dispersed phases, respectively, g./cc.
σ = surface or interfacial tension, dyne/cm.
ϕ = function plotted in Figure 3

Subscripts

D = instability solution where viscous terms are neglected
max = wave having fastest growth rate
v = instability solution where inertial terms are neglected

LITERATURE CITED

1. Christiansen, R. M., Ph.D. thesis, Univ. Pennsylvania, Philadelphia (1955).
2. ———, and A. N. Hixson, *Ind. Eng. Chem.*, **49**, 1017 (1957).
3. Meister, B. J., Ph.D. thesis, Cornell Univ., Ithaca, N. Y. (1966).
4. Merrington, R. C., and E. G. Richardson, *Proc. Phys. Soc.*, **59**, 1 (1947).
5. Middleman, Stanley, *Chem. Eng. Sci.*, **20**, 1037 (1965).
6. Plateau, M., "Statique des Liquides," Gauthier-Villars, Paris (1873).
7. Rayleigh, Lord, *Proc. London Math. Soc.*, **10**, 4 (1879).
8. ———, *Proc. Royal Soc.* (*London*), **29**, 71 (1879).
9. ———, *Phil. Mag.*, **34**, 145 (1892).
10. *Ibid.*, 177.
11. Rumscheidt, F. E., and S. G. Mason, *J. Colloid Sci.*, **17**, 260 (1962).
12. Smith, S. W. J., and H. Moss, *Proc. Royal Soc.* (*London*), **A93**, 373 (1917).
13. Tomotika, S., *ibid.*, **A150**, 322 (1935).
14. Weber, C., *Z. Angew. Math. Mech.*, **11**, 136 (1931).
15. Ziabicki, A., and R. Takserman-Krozer, *Roczniki Chem.*, **37**, 1503 (1963).
16. *Ibid.*, 38, 405 (1964).
17. ———, *Kolloid Z.*, **199**, 9 (1965).

Manuscript received December 28, 1966; revision received April 6, 1967; paper accepted April 6, 1967.